Chapter 8: Matrices

Matrix Equality:

Two matrices are equal if they have the same dimension, and the corresponding elements are equal.

Ex: Find the values of x and y, if:

$$\begin{bmatrix} 3 & \boxed{2} \\ x+y & 1 \end{bmatrix} = \begin{bmatrix} 3 & \boxed{y} \\ 2 & 1 \end{bmatrix}$$

*Solution: both matrices have 2 rows and 2 columns so they have same size

$$\begin{cases} 3 = 3 \checkmark \\ 2 = y \\ x+y = 2 \\ 1 = 1 \checkmark \end{cases}$$

$$\boxed{\begin{matrix} y = 2 \\ x = 0 \end{matrix}}$$

Square matrices

A matrix that has the same number of rows and columns is called a square matrix.

ex: $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ \sqrt{2} & 4 & \frac{3}{2} \end{bmatrix}$ $\quad$ $\begin{bmatrix} 2 \end{bmatrix}$

size: 2×2 $\quad$ 3×3 $\quad$ 1×1

A square matrix that has only nonzero entries on the diagonal and zeros everywhere else is known as a diagonal matrix.

ex: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

A special case of a diagonal matrix is the identity matrix, usually represented by I_n, where the subscript n denotes the order of the matrix.

Ex: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

A square matrix with all its entries being zero, is known as the null matrix.

Exercises:

1) Find the values of x and y if: $\begin{bmatrix} 1 & 2 \\ x-y & 2 \end{bmatrix} = \begin{bmatrix} 1 & y \\ 0 & 2 \end{bmatrix}$

$\left.\begin{matrix} y = 2 \\ x - y = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x = 2 \\ y = 2 \end{matrix}$

2) Same when $\begin{bmatrix} x & y \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ Not possible

these matrices can not be equal

Basix Matrix Operations

* Addition and Subsbraction of Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$ can not be done; they have to have same size

* Scalar multiplication

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad 3A = \begin{bmatrix} 6 & 3 \\ 9 & 6 \end{bmatrix} \qquad kA = \begin{bmatrix} 2k & 1k \\ 3k & 2k \end{bmatrix}$$

Ex: Car Production

A car manufacturer who produces three different models in 3 different plants A, B, and C reaches the production levels in million of dollars in the first and second half of the year as follows:

First half

	Model 1	Model 2	Model 3
Plant A	27	44	51
Plant B	35	39	62
Plant C	33	50	47

Second half

	Model 1	Mod. 2	Mod 3
Plant A	25	42	48
Plant B	33	40	66
Plant C	35	48	50

Total production by each plant for the whole year is.

$$\begin{bmatrix} 27 & 44 & 51 \\ 35 & 39 & 62 \\ 33 & 50 & 47 \end{bmatrix} + \begin{bmatrix} 25 & 42 & 48 \\ 33 & 40 & 66 \\ 35 & 48 & 50 \end{bmatrix} =$$

$$\begin{bmatrix} 52 & 86 & 99 \\ 68 & 79 & 128 \\ 68 & 98 & 97 \end{bmatrix}$$

Matrix multiplication

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + 2y \\ -x + 4y \end{bmatrix}$$

$2 \times \cancel{2}$ $\quad \cancel{2} \times 1$ $\quad 2 \times 1$

1st row x 1st column ⟹ answer will be in 1st row, 1st column.

2nd row x 1st column ⟹ answer will be in 2nd row, 1st column.

$$\begin{bmatrix}2 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}4 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}8 & 5\\ -4 & -2\end{bmatrix}$$

$2 \times \cancel{2} \quad \cancel{2} \times 2 \qquad\qquad 2 \times 2$

have to be same

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -5 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$2 \times \cancel{3}$ $\quad$ $\cancel{3} \times 4$

$$= \begin{bmatrix} 4 & 2 & -9 & 0 \\ -2 & -2 & -2 & 4 \end{bmatrix}$$

2×4

Ex: Cost of a Basket of Goods

Let the column vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ denote the quantities of n goods that a consumer buys in a week. Let the row vector $p = [p_1 \;\; p_2 \;\; \cdots \;\; p_n]$ denote the corresponding prices in dollars of a unit of each good (p_1 is the price of one unit of x_1, p_2 is the price of one unit of x_2, $\ldots$)

Find the consumer's weekly expenditure on these ~~goo~~ goods.

<u>Solution:</u>

The consumer's weekly expenditure on goods is given by:

$$E = \underbrace{\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}}_{1 \times \cancel{n}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\cancel{n} \times 1}$$

$$= \underbrace{\begin{bmatrix} p_1x_1 + p_2x_2 + \cdots + p_nx_n \end{bmatrix}}_{1 \times 1}$$

<u>Remark</u>: the ~~matri~~ product matrix AB is well defined if the number of columns of A is the same as the number of rows in B

<u>Def</u>: The matrix A^n is the product matrix obtained by multiplying the square matrix A by itself n times

!!! a square matrix is a matrix that has same number of rows and columns

Exercises :

1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Find $3A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2) a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}$

Find $A - B =$ not possible (A & B don't have same size)

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Find $A - B = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

4) a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $2 \times \cancel{3}$ $\quad B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$ $\cancel{3} \times 2$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$$

Extra Problem. Let $X^0 = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$ be the populations of 3 regions in millions at some initial point in time, 0. The transition matrix is given by: $P = \begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.10 & 0.70 & 0.05 \\ 0.10 & 0.15 & 0.90 \end{bmatrix}$

Find the population of the 3 regions at the beginning of the next period x^1.

Solution: ~~the~~ we have to solve the equation

$$X^1 = P X^0,$$

$$X^1 = \begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.10 & 0.70 & 0.05 \\ 0.10 & 0.15 & 0.90 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5.8 \\ 7.8 \\ 7.4 \end{bmatrix}$$

After 2^{nd} period: $P \cdot X^1 = X^2$

After 3^{rd} period: $P \cdot X^2 = X^3$

Def: the <u>transpose matrix</u>, A^T, is the original matrix A with its rows and columns interchanged

ex: $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 5 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 5 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 4 \\ 1 & 4 & -1 \end{bmatrix}$

$B^T = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 4 \\ 1 & 4 & -1 \end{bmatrix}$

$B = B^T$ because B is symmetric.

<u>Def</u>: A matrix A is equal to its transpose A^T is called a <u>symmetric matrix</u>.

<u>Theorem</u>: $(A+B)^T = A^T + B^T$

$$(AB)^T \neq A^T B^T \quad !!!$$

but $(AB)^T = B^T A^T$

Exercises:

1) a) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad I_3^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 7 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

a) $-3A = \begin{bmatrix} -21 & 0 & 3 \\ -6 & -9 & -3 \end{bmatrix}$

b) $A + E =$ not possible

c) $B - 3D =$ not possible

d) $3C - E =$ not possible

e) $AC = \cancel{(2\times 3)(2\times 2)}$ not possible

3) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$

$(AB)^T = ?$

$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}$

Def: A square matrix A of any order is <u>idempotent</u> if $A = A^2 = A^3 = \dots$

Def: The <u>trace</u> of a square matrix A is given by the sum of the elements of the main diagonal

<u>ex:</u> $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & -1 \\ 4 & 2 & 5 \end{bmatrix}$ $\quad tr(A) = 2+0+5 = 7$

Exercises: p. 297

1) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ I is idempotent?

$$I^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

so I is idempotent

3) $A = \begin{bmatrix} x & -x \\ x-1 & 1-x \end{bmatrix}$ A idempotent?

$$A^2 = \begin{bmatrix} x & -x \\ x-1 & 1-x \end{bmatrix} \begin{bmatrix} x & -x \\ x-1 & 1-x \end{bmatrix}$$

$2 \times \cancel{2}$ $\cancel{2} \times 2$

$$= \begin{bmatrix} x^2+(-x)(x-1) & -x^2+(-x)(1-x) \\ x(x-1)+(1-x)(x-1) & -x(x-1)+(1-x)^2 \end{bmatrix} = \begin{bmatrix} x & -x \\ x-1 & 1-x \end{bmatrix} = A$$

2×2

$A^2 = A$, so A is idempotent

Remark: $A = A^2$

I multiply both sides by A

$$A . A = A^2 . A$$

$$A^2 = A^3$$

but $A^2 = A \Rightarrow A = A^2 = A^3$ etc...

So it should be enough to prove that $A = A^2$, to say that A is idempotent.

4) $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

Prove that $\boxed{tr(AB) = tr(BA)}$

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 6 & 4 \end{bmatrix}$$

$tr(AB) = 4 + 4 = 8$

$$BA = \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 3 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

$tr(BA) = 5 + 2 + 1 = 8$